Representing and Manipulating Integers

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Examples

```
int i = 1;
short j = 2;
i=(int) j;
j=(short) i;
```

```
int i = 70000;
short j = 2;
j=(short) i;
```

```
int i = -5;
unsigned k = 6;
i=(int) k;
k=(unsigned) i;
```

```
int i = -5;
unsigned k = -6;
i=(int) k;
k=(unsigned) i;
```

Examples

1. `char -> short`
2. `short -> int`
3. `int -> long`
4. `long -> int`
5. `int -> short`
6. `short -> char`
7. `unsigned -> signed`
8. `signed -> unsigned`

Unsigned Integers

Encoding unsigned integers

$$B = [b_{w-1}, b_{w-2}, \ldots, b_0]$$

$$D(B) = \sum_{i=0}^{w-1} b_i \cdot 2^i$$

x = 0000 0111 1101 0011$_2$

$$\begin{aligned} D(x) &= 2^{10} + 2^9 + 2^8 + 2^7 \\ &\quad + 2^6 + 2^4 + 2^1 + 2^0 \\ &= 1024 + 512 + 256 + 128 \\ &\quad + 64 + 16 + 2 + 1 \\ &= 2003 \end{aligned}$$

What is the range for unsigned values with w bits?

Signed Integers (1)

Encoding positive numbers

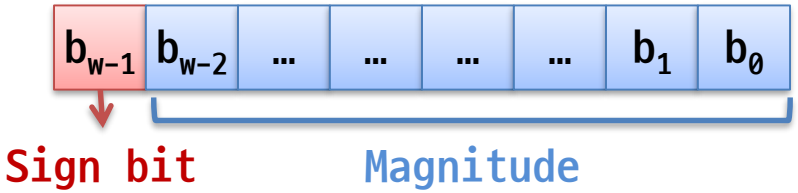
- Same as unsigned numbers

Encoding negative numbers

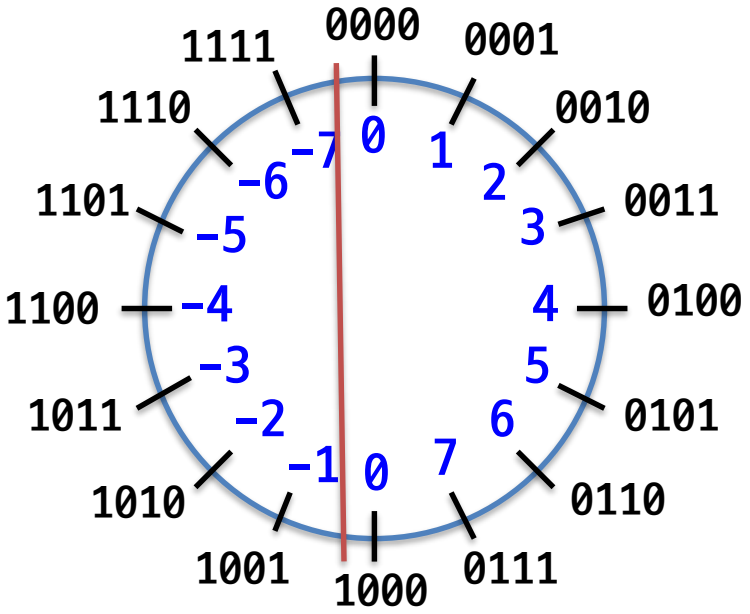
- Sign-magnitude representation
- Ones' complement representation
- Two's complement representation

Signed Integers (2)

Sign-magnitude representation



$$S(B) = (-1)^{b_{w-1}} \cdot \left(\sum_{i=0}^{w-2} b_i \cdot 2^i \right)$$

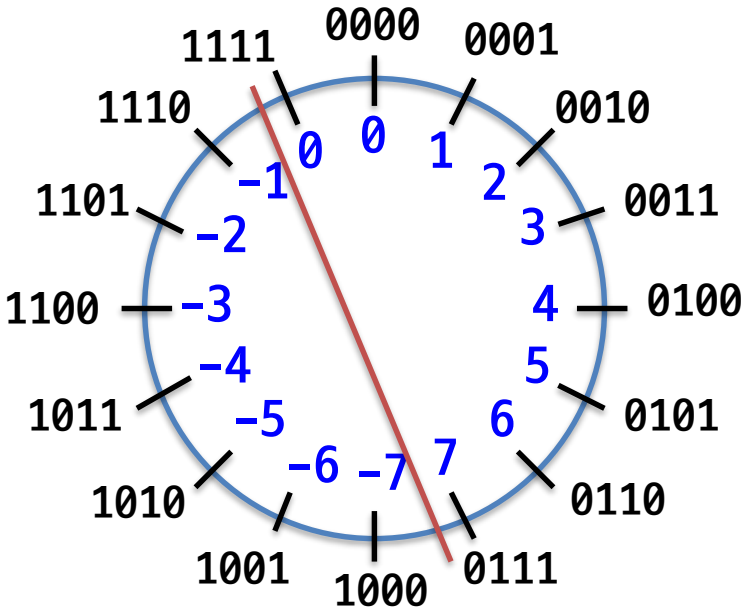


- Two zeros
  - [000…00], [100…00]
- Used for floating-point numbers

Signed Integers (3)

Ones' complement representation

$$O(B) = -b_{w-1}(2^{w-1} - 1) + \left(\sum_{i=0}^{w-2} b_i \cdot 2^i \right)$$

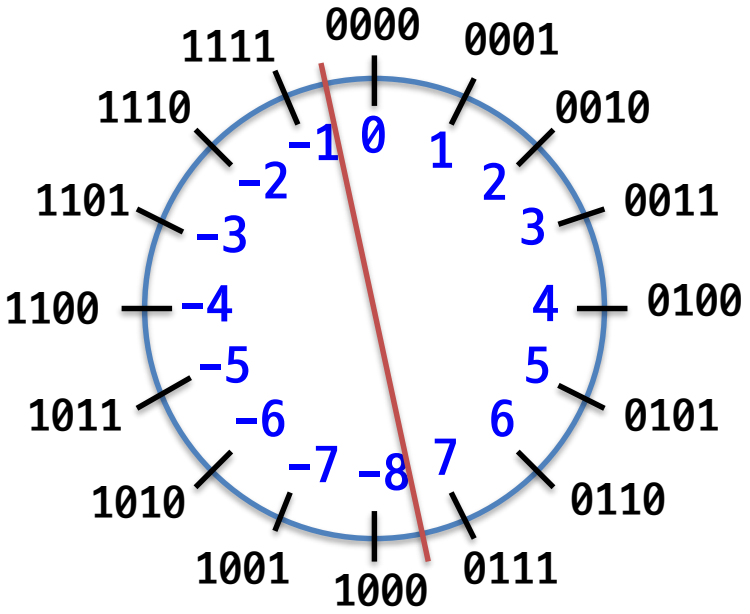


- Easy to find -n
- Two zeros
  - [000…00], [111…11]
- No longer used

Signed Integers (4)

Two's complement representation

$$O(B) = -b_{w-1} \cdot 2^{w-1} + \left(\sum_{i=0}^{w-2} b_i \cdot 2^i \right)$$



- Unique zero
- Easy for hardware
  - leading 0 ≥ 0
  - leading 1 < 0
- Used by almost all modern machines

Representation of Negative Numbers

$+N$	Positive Integers (all systems)	$-N$	Negative Integers		
			Sign and Magnitude	2's Complement N^*	1's Complement $\bar{N}$
+0	0000	−0	1000	——	1111
+1	0001	−1	1001	1111	1110
+2	0010	−2	1010	1110	1101
+3	0011	−3	1011	1101	1100
+4	0100	−4	1100	1100	1011
+5	0101	−5	1101	1011	1010
+6	0110	−6	1110	1010	1001
+7	0111	−7	1111	1001	1000
		−8	——	1000	——

Signed Integers (5)

Two's complement representation (cont'd)

- Following holds for two's complement

```
~ x  +  1  ==  - x
```

- Complement
  - Observation: ~x + x == 1111…11_2 == -1
- Increment

  ~x + x == -1

  ~x + x + (-x + 1) == -1 + (-x + 1)

  ~x + 1 == -x

Numeric Ranges (1)

Unsigned values

- UMin = 0 [000…00]
- UMax = $2^w - 1$ [111…11]

Two's complement values

- TMin = -2^{w-1} [100…00]
- TMax = $2^{w-1} - 1$ [011…11]

Values for w = 16

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Numeric Ranges (2)

Values for different word sizes

	w = 8	w = 16	w = 32	w = 64
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

Observations

- ¦TMin¦ = TMax + 1
  (Asymmetric range)
- UMax = 2 * TMax + 1

In C programming

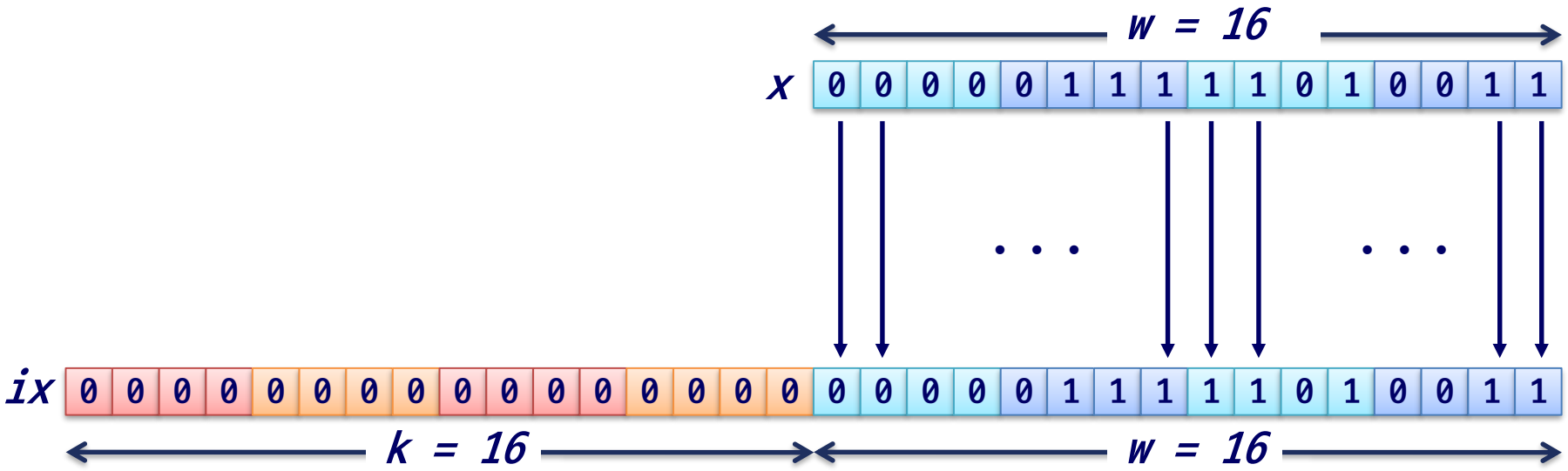
- #include <limits.h>
- INT_MIN, INT_MAX,
  LONG_MIN, LONG_MAX,
  UINT_MAX, …
- Values platform-specific

Type Conversion (1)

Unsigned: *w* bits -> *w+k* bits

- Zero extension: just fill *k* bits with 0's

```
unsigned short x  = 2003;
unsigned       ix = (unsigned) x;
```



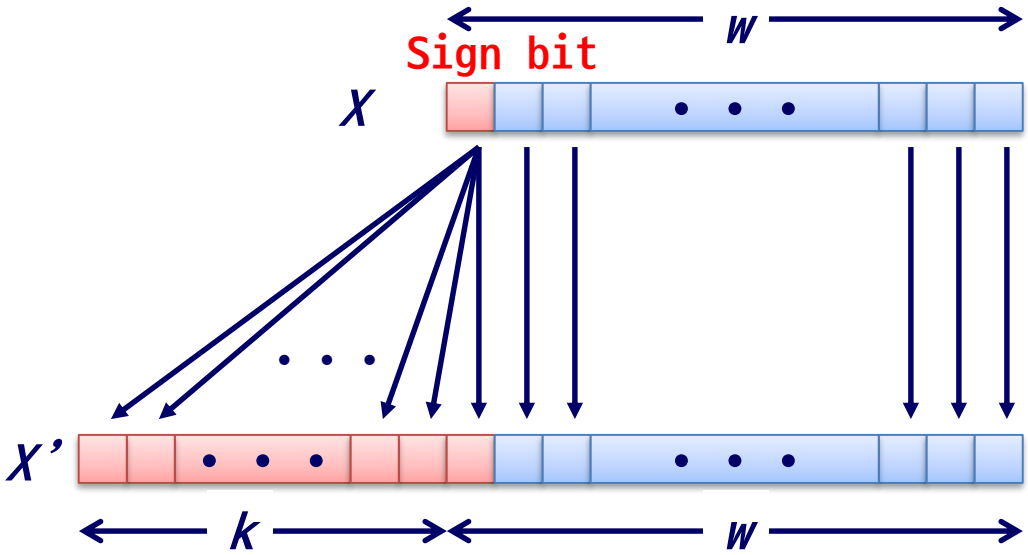
Type Conversion (2)

Signed: *w* bits -> *w+k* bits

- Given *w*-bit signed integer *x*
- Convert it to *w+k* bit integer with same value

Sign extension

- Make *k* copies of sign bit



Type Conversion (3)

Sign extension example

- Converting from smaller to larger integer type
- C automatically performs sign extension

```
short  int  x    =  2003;
       int  ix   =  (int) x;
short  int  y    =  -2003;
       int  iy   =  (int) y;
```

	Decimal	Hex	Binary
x	2003	07 D3	00000111 11010011
ix	2003	00 00 07 D3	00000000 00000000 00000111 11010011
y	-2003	F8 2D	11111000 00101101
iy	-2003	FF FF F8 2D	11111111 11111111 11111000 00101101

Type Conversion (4)

Unsigned & Signed: *w+k* bits -> *w* bits

- Just truncate it to lower *w* bits
- Equivalent to computing *x* mod 2^w

```
unsigned  int    x   =  0xcafebabe;
unsigned  short  ix  =  (unsigned short) x;
          int    y   =  0x2003beef;
          short  iy  =  (short) y;
```

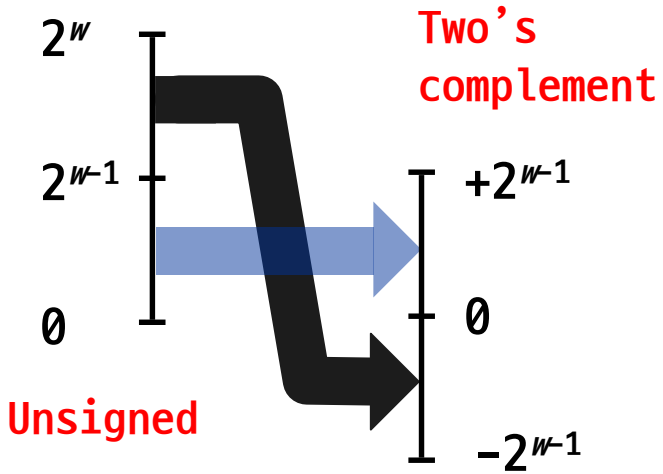
	Decimal	Hex	Binary
x	3405691582	CA FE BA BE	11001010 11111110 10111010 10111110
ix	47806	BA BE	10111010 10111110
y	537116399	20 03 BE EF	00100000 00000011 10111110 11101111
iy	-16657	BE EF	10111110 11101111

Type Conversion (5)

Unsigned -> Signed

- The same bit pattern is interpreted as a signed number

$$U2T_w(x) = \begin{cases} x, & x < 2^{w-1} \\ x - 2^w, & x \geq 2^{w-1} \end{cases}$$




```
unsigned short x  = 2003;
         short ix = (short) x;
unsigned short y  = 0xbabe;
         short iy = (short) y;
```

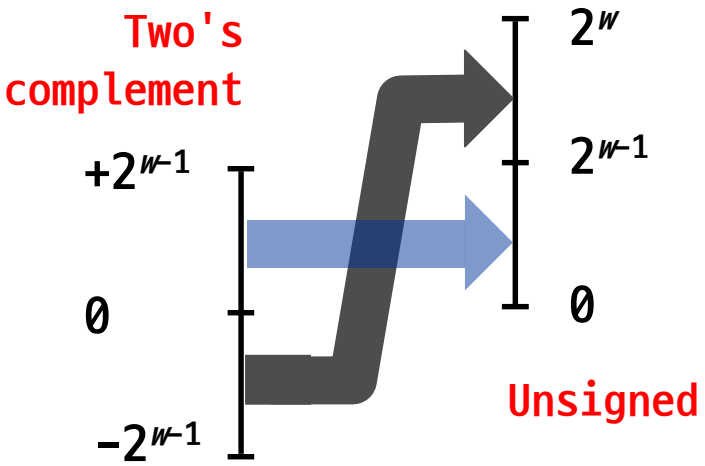
	Decimal	Hex	Binary
x	2003	07 D3	00000111 11010011
ix	2003	07 D3	00000111 11010011
y	47806	BA BE	10111010 10111110
iy	-17730	BA BE	10111010 10111110

Type Conversion (6)

Signed -> Unsigned

- Ordering inversion
- Negative -> Big positive

$$T2U_w(x) = \begin{cases} x + 2^w, & x < 0 \\ x, & x \geq 0 \end{cases}$$




```
          short x  = 2003;
unsigned short ix = (unsigned short) x;
          short y  = -2003;
unsigned short iy = (unsigned short) y;
```

	Decimal	Hex	Binary
x	2003	07 D3	00000111 11010011
ix	2003	07 D3	00000111 11010011
y	-2003	F8 2D	11111000 00101101
iy	63533	F8 2D	11111000 00101101

Type Casting in C (1)

Constants

- By default, considered to be signed integers
- Unsigned if have "U" or "u" as suffix
  - 0U, 12345U, 0x1A2Bu

Type casting

- Explicit casting

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

- Implicit casting via
  - Assignments
  - Procedure calls

```
int f(unsigned);
f(ty);
tx = ux;
```

Type Casting in C (2)

Expression evaluation

- If mix unsigned and signed in single expression, signed values implicitly cast to unsigned
- Including comparison operations <, >, ==, <=, >=

Expression	Type	Evaluation
0 == 0U		
-1 < 0		
-1 < 0U		
-1 > -2		
(unsigned) -1 > -2		
2147483647 > -2147483647-1		
2147483647U > -2147483647-1		
2147483647 > (int) 2147483648U		

Type Casting in C (2) – appendix

Compiler converts to signed int for short, char

```
#include <stdio.h>

int main()
{
    int a=-1;
    unsigned int b=2;
    short c=-1;
    unsigned short d=2;
    long e=-1;
    unsigned long f=2;
    long long g=-1;
    unsigned long long h=2;
    char i=-1;
    unsigned char j=2;

    if (a>b)
        printf("a>b\n");
    else
        printf("a>b ???\n");
    if (c>d)
        printf("c>d\n");
    else
        printf("c>d ???\n");
    if (e>f)
        printf("e>f\n");
    else
        printf("e>f ???\n");

    if (g>h)
        printf("g>h\n");
    else
        printf("g>h ???\n");

    if (i>j)
        printf("i>j\n");
    else
        printf("i>j ???\n");
}
```

```
a>b
c>d ???
e>f
g>h
i>j ???
```

Type Casting in C (2) – appendix

Compiler converts to signed int for short, char

```
#include<stdio.h>

int main()
{
    signed short b = -10;
    unsigned short c = -10;
    signed int a = -10;
    unsigned int d = -10;

    printf("%d , %d\n", b, (unsigned short)b);
    printf("%d , %d\n", (signed short)c, c);
    printf("%d , %u\n", a, (unsigned int)a);
    printf("%d , %u\n", (signed int)d, d);
    printf("b==c %d\n", b==c);
    printf("a==d %d\n", a==d);
    return 0;
}
```

```
-10 , 65526
-10 , 65526
-10 , 4294967286
-10 , 4294967286
b==c 0
a==d 1
```

Type Casting in C (3)

Example 1-1

```
int  main  ()
{
     unsigned  i;
     for  (i = 10;  i >= 0;  i--)
          printf  ("%u\n",  i);
}
```

Example 1-2

```
int  main  ()
{
     unsigned  i;
     for  (i = 10;  i > 0;  i--)
          printf  ("%u\n",  i);
}
```

Type Casting in C (4)

Example 2

```
int  sum_array  (int  a[],  unsigned  len)
{
     int  i;
     int  result = 0;

     for  (i  =  0;  i  <=  len  -  1;  i++)
          result  +=  a[i];

     return  result;
}
```

Type Casting in C (5)

Example 3-1

```
void copy_mem1 (char *src, char *dest, unsigned len)
{
    unsigned i;
    for (i = 0; i < len; i++)
        *dest++ = *src++;
}
```

Example 3-2

```
void copy_mem2 (char *src, char *dest, unsigned len)
{
    int i;
    for (i = 0; i < len; i++)
        *dest++ = *src++;
}
```

Type Casting in C (6)

Example 3-3

```
void copy_mem3 (char *src, char *dest, unsigned len)
{
    for (; len > 0; len--)
        *dest++ = *src++;
}
```

Example 3-4

```
void copy_mem4 (char *src, char *dest, unsigned len)
{
    for (; (int) len > 0; len--)
        *dest++ = *src++;
}
```

Type Casting in C (7)

Example 4

```
#include <stdio.h>

int  main  ()
{
     unsigned char  c;

     while  ((c  =  getchar())  !=  EOF)
          putchar  (c);
}
```

Type Casting in C (8)

Lessons

- There are many tricky situations when you use unsigned integers – hard to debug
- Do not use just because numbers are nonnegative
- Use "unsigned" for large positive numbers within limited bits
- Use "unsigned" when you need collections of bits with no numeric interpretation ("flags")
- Few languages other than C support unsigned integers

Bit-Level Operations in C

Operations &, ¦, ~, ^ available in C

- Apply to any "integral" data type
  - long, int, short, char, unsigned
- View arguments as bit vectors
- Arguments applied bit-wise

Examples (Char data type)

- ~0x41 --> 0xBE
- $\sim 01000001_2$ --> 10111110_2
- ~0x00 --> 0xFF
- $\sim 00000000_2$ --> 11111111_2
- 0x69 & 0x55 --> 0x41
- 01101001_2 & 01010101_2 --> 01000001_2
- 0x69 ¦ 0x55 --> 0x7D
- 01101001_2 ¦ 01010101_2 --> 01111101_2
- 0x69 ^ 0x55 --> 0x3C
- 01101001_2 ^ 01010101_2 --> 00111100_2

Logic Operations in C

Logical operators

- &&, ||, !
- View 0 as "False"
- Anything nonzero as "True"
- Always return 0 or 1

Early termination

```
int i=1;
int a=4, b=4, c=5;

if ( a == b && i > c )
{
        do_something();
}
```

```
int i=1;
int a=4, b=4, c=5;

if ( i > c && a == b )
{
        do_something();
}
```

Logic Operations in C

Examples (char data type)

- !0x41 --> 0x00
- !0x00 --> 0x01
- !!0x41 --> 0x01

- 0x69 && 0x55 -->
- 0x69 ¦¦ 0x55 -->

```
if (p && *p) // avoids null pointer access
{
    // p is not null pointer && *p is not 0
}
```

Shift Operations

Left shift: **x << y**

- Shift bit-vector x left y positions
  - Throw away extra bits on left
  - Fill with 0's on right

Argument x	01100010
<< 3	00010000
Log. >> 2	00011000
Arith. >> 2	00011000

Right shift: **x >> y**

- Shift bit-vector x right y positions
  - Throw away extra bits on right
- Logical shift
  - Fill with 0's on left
- Arithmetic shift
  - Replicate MSB on right
  - Useful with two's complement integer representation

Argument x	10100010
<< 3	00010000
Log. >> 2	00101000
Arith. >> 2	11101000

Undefined if y < 0 or y ≥ word size

Shift Operations

Right shift example

```
void main()
{
    char i = -2;

    printf("%d\n", i);
    i = i >> 2;
    printf("%d\n", i);
}
```

"Arithmetic shift right" is the default operation in C

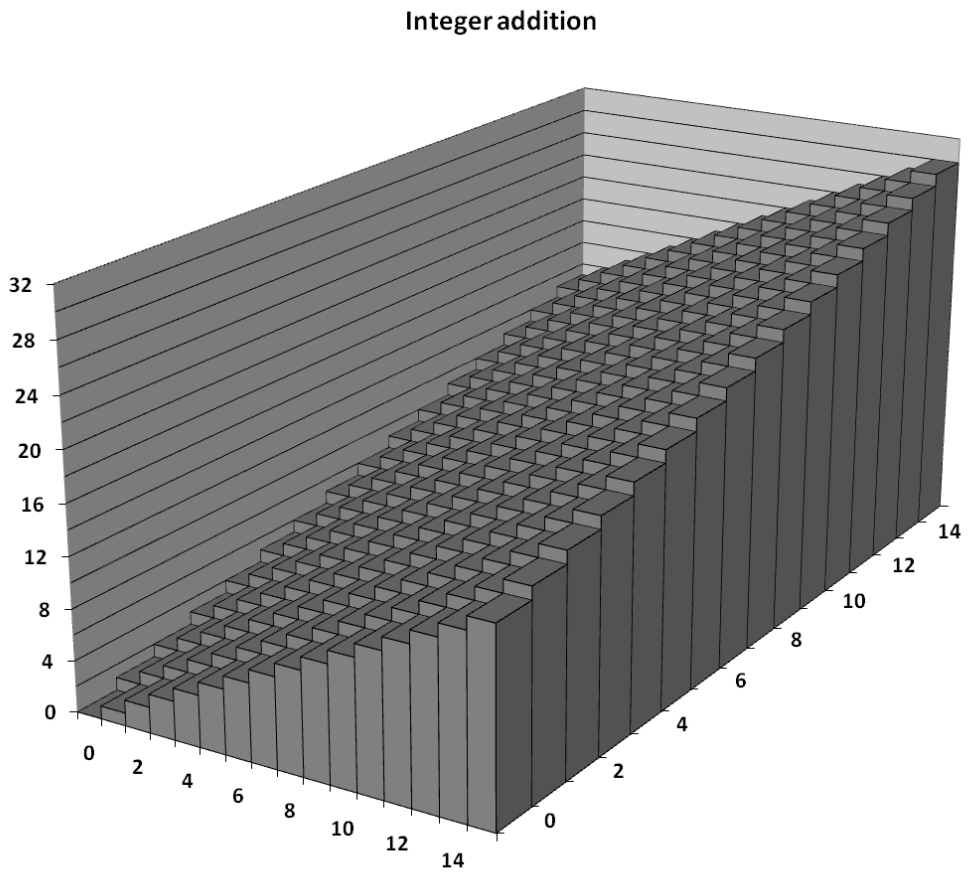
```
void main()
{
    char i = -2;

    printf("%d\n", i);
    i = i >> -2;
    printf("%d\n", i);
}
```

Addition (1)

Integer addition example

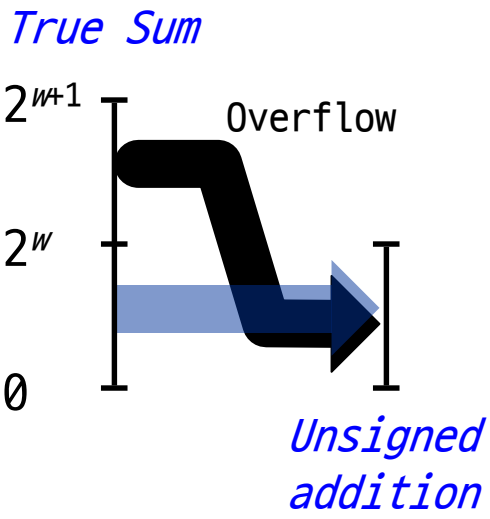
- 4-bit integers *u*, *v*
- Compute true sum
- True sum requires one more bit ("carry")
- Values increase linearly with *u* and *v*
- Forms planar surface

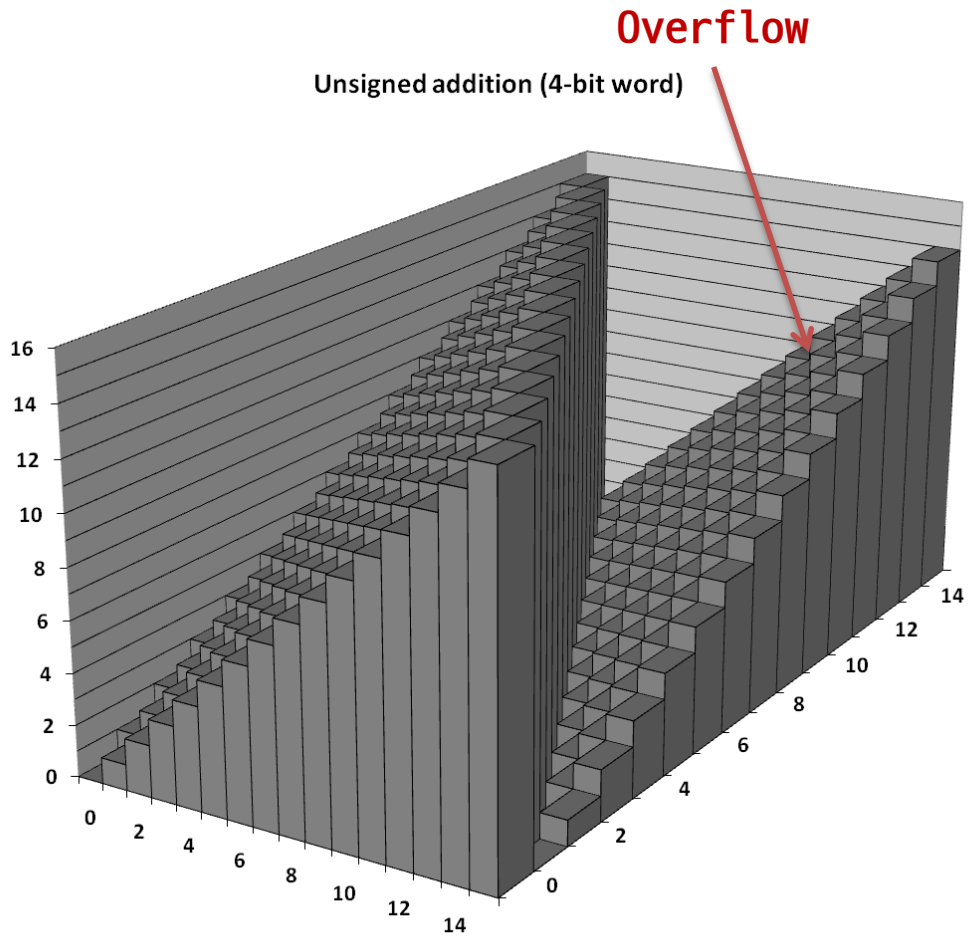


Addition (2)

Unsigned addition

- Ignores carry output
- Wraps around
  - If true sum ≥ 2^w

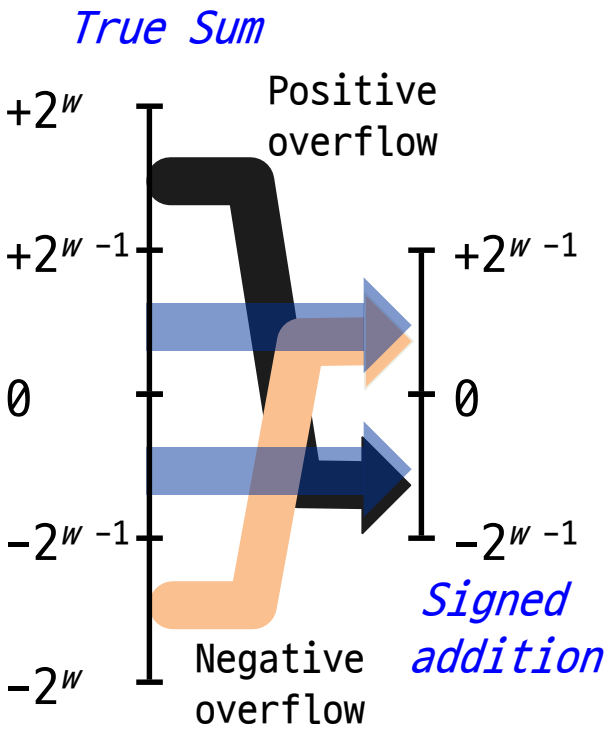


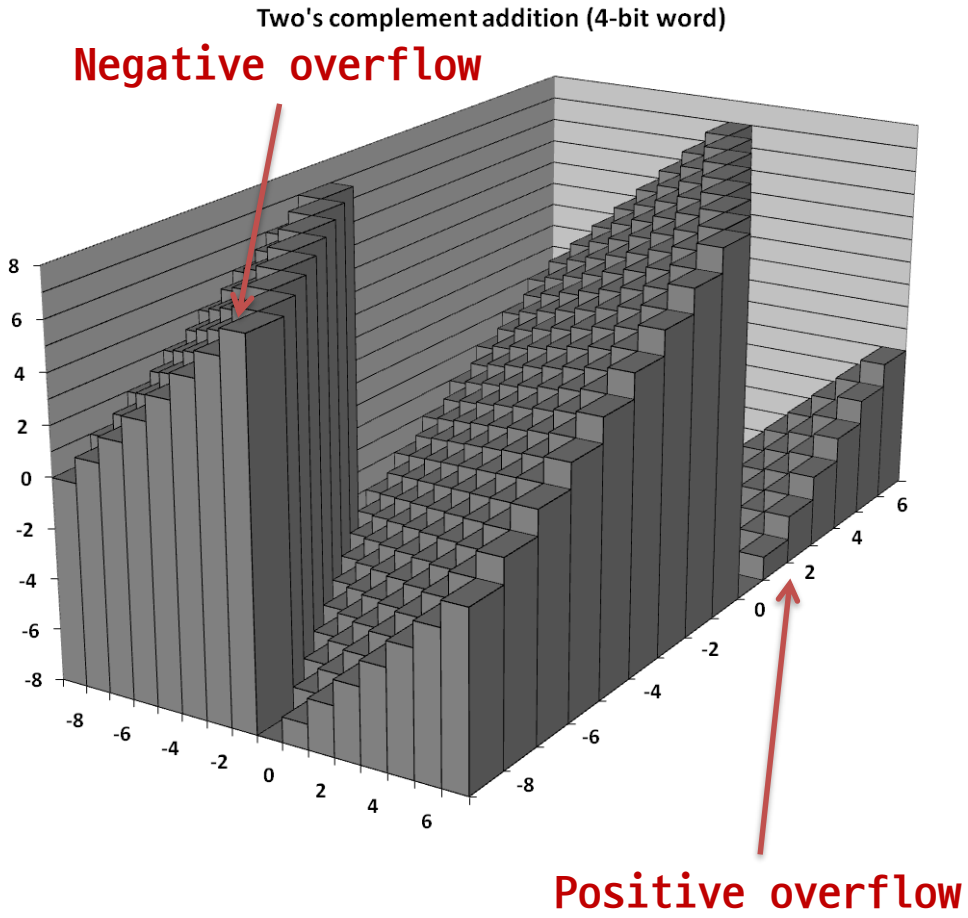


Addition (3)

Signed addition

- Drop off MSB
- Treat remaining bits as 2's comp. integer





Addition (4)

Signed addition in C

- Ignores carry output
- The low-order w bits are identical to unsigned addition

Examples for w = 3

Mode	x	y	x + y	Truncated x + y
Unsigned	4 [100]	3 [011]	7 [0111]	7 [111]
Two's comp.	-4 [100]	3 [011]	-1 [1111]	-1 [111]
Unsigned	4 [100]	7 [111]	11 [1011]	3 [011]
Two's comp.	-4 [100]	-1 [111]	-5 [1011]	3 [011]
Unsigned	3 [011]	3 [011]	6 [0110]	6 [110]
Two's comp.	3 [011]	3 [011]	6 [0110]	-2 [110]

Multiplication (1)

Ranges of (x * y)

- Unsigned: up to 2w bits

  $$0 \le x * y \le (2^w - 1)^2 \;=\; 2^{2w} - 2^{w+1} + 1$$

- Two's complement min: up to 2w–1 bits

  $$x * y \;\ge\; (-2^{w-1})*(2^{w-1}-1) \;=\; -2^{2w-2} + 2^{w-1}$$

- Two's complement max: up to 2w bits (only for $TMin^2$)

  $$x * y \le (-2^{w-1})^2 \;=\; 2^{2w-2}$$

Maintaining exact results

- Would need to keep expanding word size with each product computed
- Done in software by "arbitrary precision" arithmetic packages

Multiplication (2)

Unsigned multiplication in C

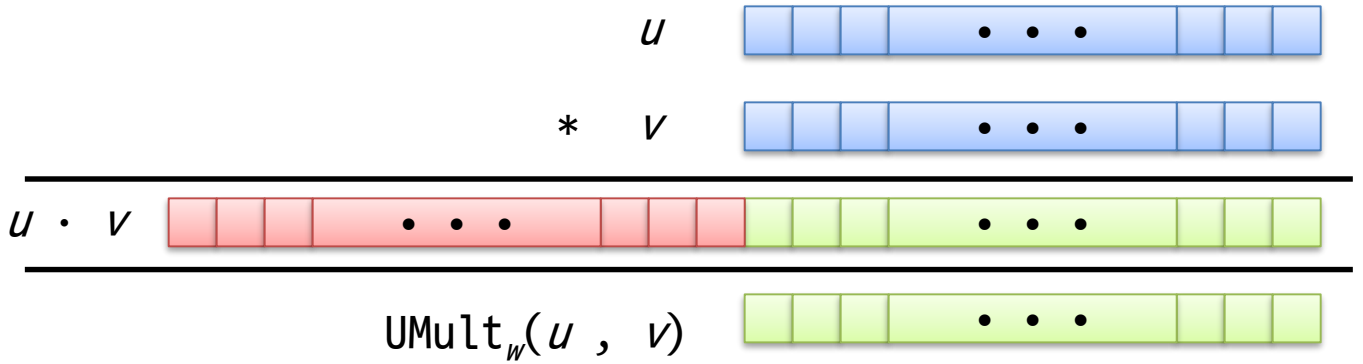
- Ignores high order w bits
- Implements modular arithmetic

$$UMult_w(u, v) = u \cdot v \mod 2^w$$

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



Multiplication (3)

Signed multiplication in C

- Ignores high order w bits
- The low-order w bits are identical to unsigned multiplication

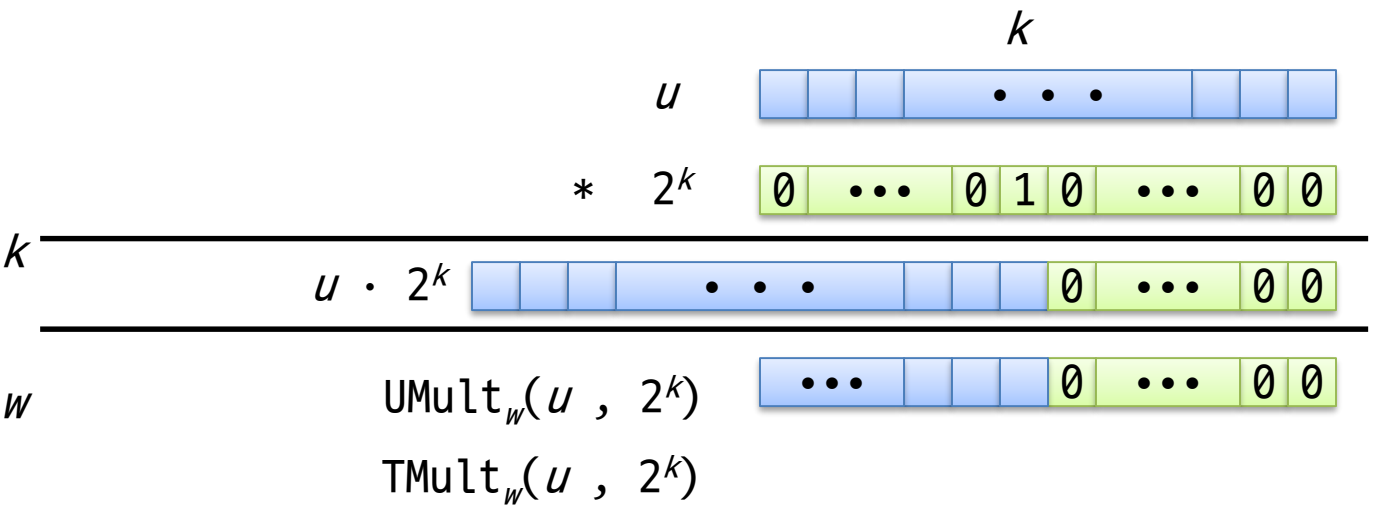
Examples for w = 3

Mode	x	y	x · y	Truncated x · y
Unsigned	5 [101]	3 [011]	15 [001111]	7 [111]
Two's comp.	-3 [101]	3 [011]	-9 [110111]	-1 [111]
Unsigned	4 [100]	7 [111]	28 [011100]	4 [100]
Two's comp.	-4 [100]	-1 [111]	4 [000100]	-4 [100]
Unsigned	3 [011]	3 [011]	9 [001001]	1 [001]
Two's comp.	3 [011]	3 [011]	9 [001001]	1 [001]

Multiplication (4)

Power-of-2 multiply with shift

- *u << k* gives *u * 2^k*
  - e.g., *u << 3 == u * 8*
- Both signed and unsigned
- Most machines shift and add faster than multiply
  - Compiler generates this code automatically



Multiplication (5)

Compiled multiplication code

- C compiler automatically generates shift/add code when multiplying by constant

C Function

```
int mul12 (int  x)
{
   return x * 12;
}
```

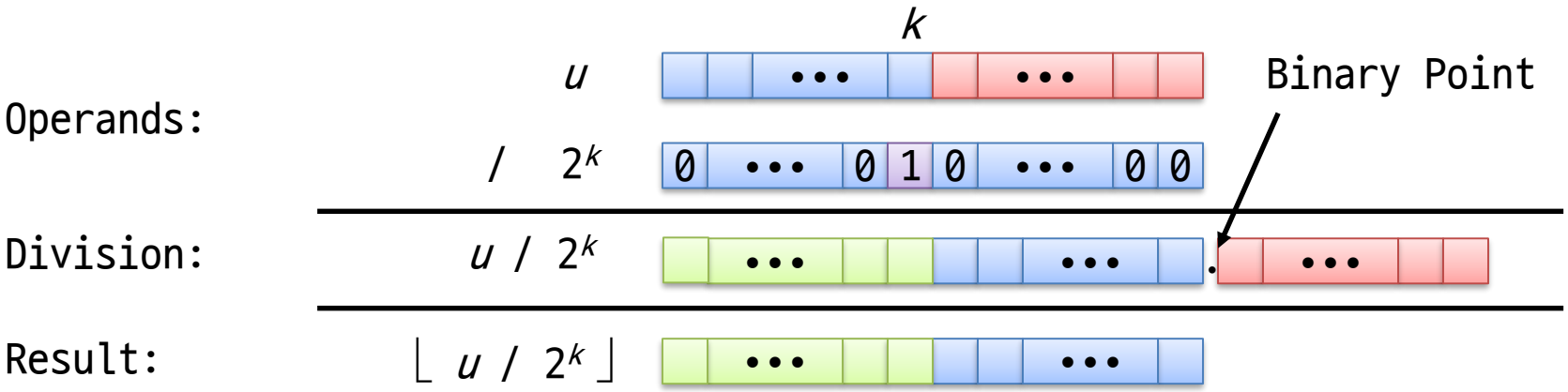
Compiled Arithmetic Operations

```
leal   (%eax, %eax, 2), %eax          ;  t ← x + x * 2
sall   $2, %eax                       ;  return  t << 2
```

Division (1)

Unsigned power-of-2 divide with shift (including signed positive)

- $u >> k$ gives $\lfloor u / 2^k \rfloor$
- Uses logical shift



Expression	Division	Result	Hex	Binary
x	15213	15213	3B 6D	00111011 01101101
x >> 1	7606.5	7606	1D B6	00011101 10110110
x >> 4	950.8125	950	03 B6	00000011 10110110
x >> 8	59.4257813	59	00 3B	00000000 00111011

Division (2)

Compiled unsigned division code

- Uses logical shift for unsigned
- Logical shift written as >>> in Java

C Function

```
unsigned udiv8 (unsigned x)
{
    return x / 8;
}
```

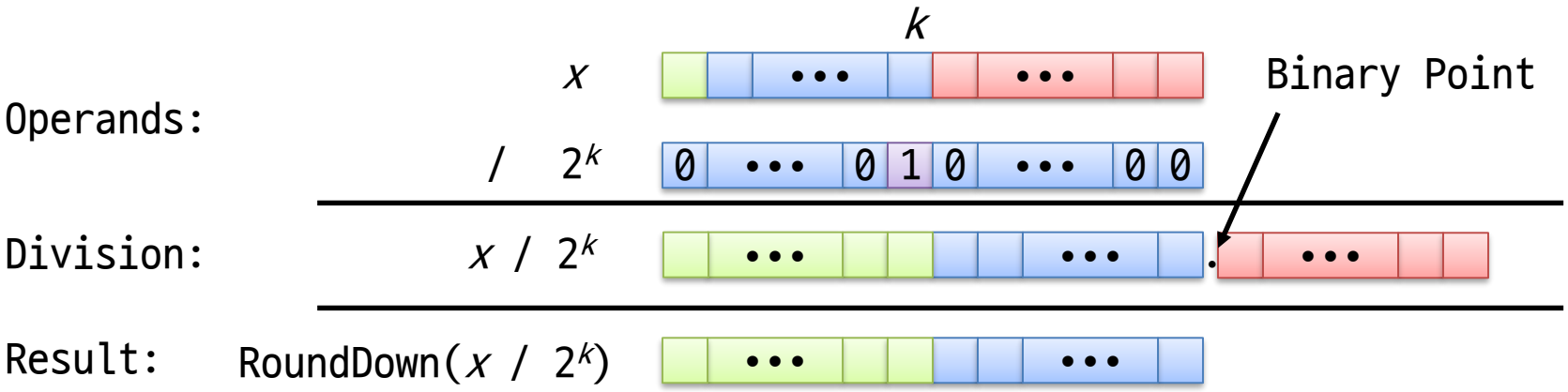
Compiled Arithmetic Operations

```
shrl    $3, %eax                        ;  return t >> 3
```

Division (3)

Signed power-of-2 divide with shift

- $x >> k$ gives $\lfloor x / 2^k \rfloor$
- Uses arithmetic shift (rounds wrong direction if $x < 0$)



Expression	Division	Result	Hex	Binary
y	-15213	-15213	C4 93	11000100 10010011
y >> 1	-7606.5	-7607	E2 49	11100010 01001001
y >> 4	-950.8125	-951	FC 49	11111100 01001001
y >> 8	-59.4257813	-60	FF C4	11111111 11000100

Division (4)

Correct power-of-2 divide

- Want $\lceil x / 2^k \rceil$ (Round Toward 0) when $x < 0$
- Compute as $\lceil (x + 2^k - 1) / 2^k \rceil$
  - In C: `(x + (1 << k) - 1) >> k`
  - Biases dividend toward 0

Division (4)

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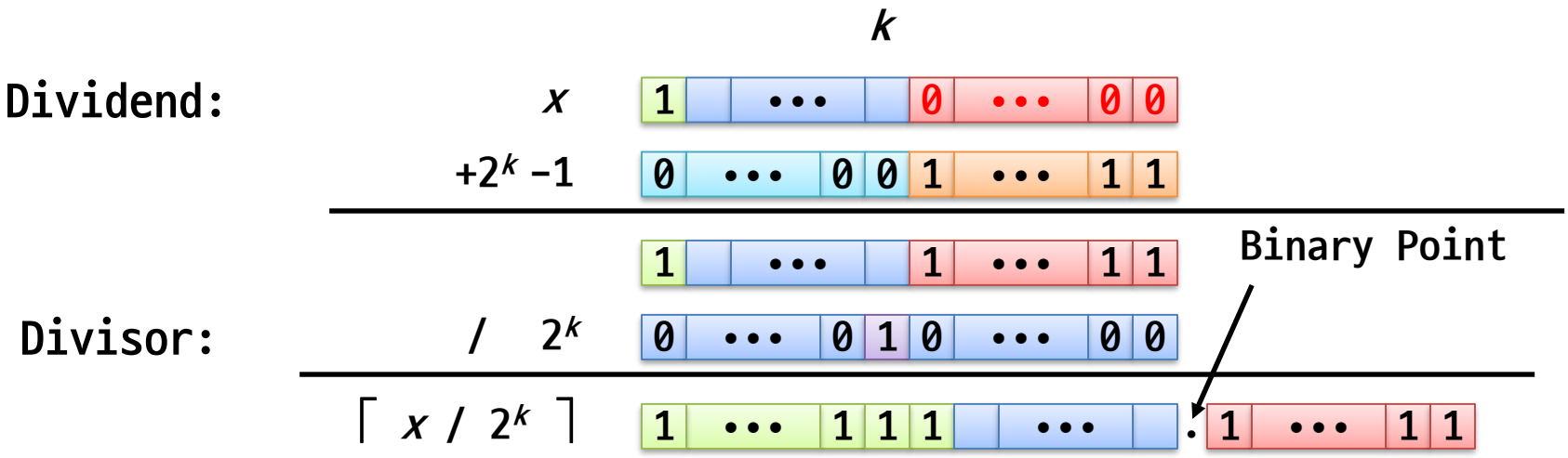
Case 1: No rounding

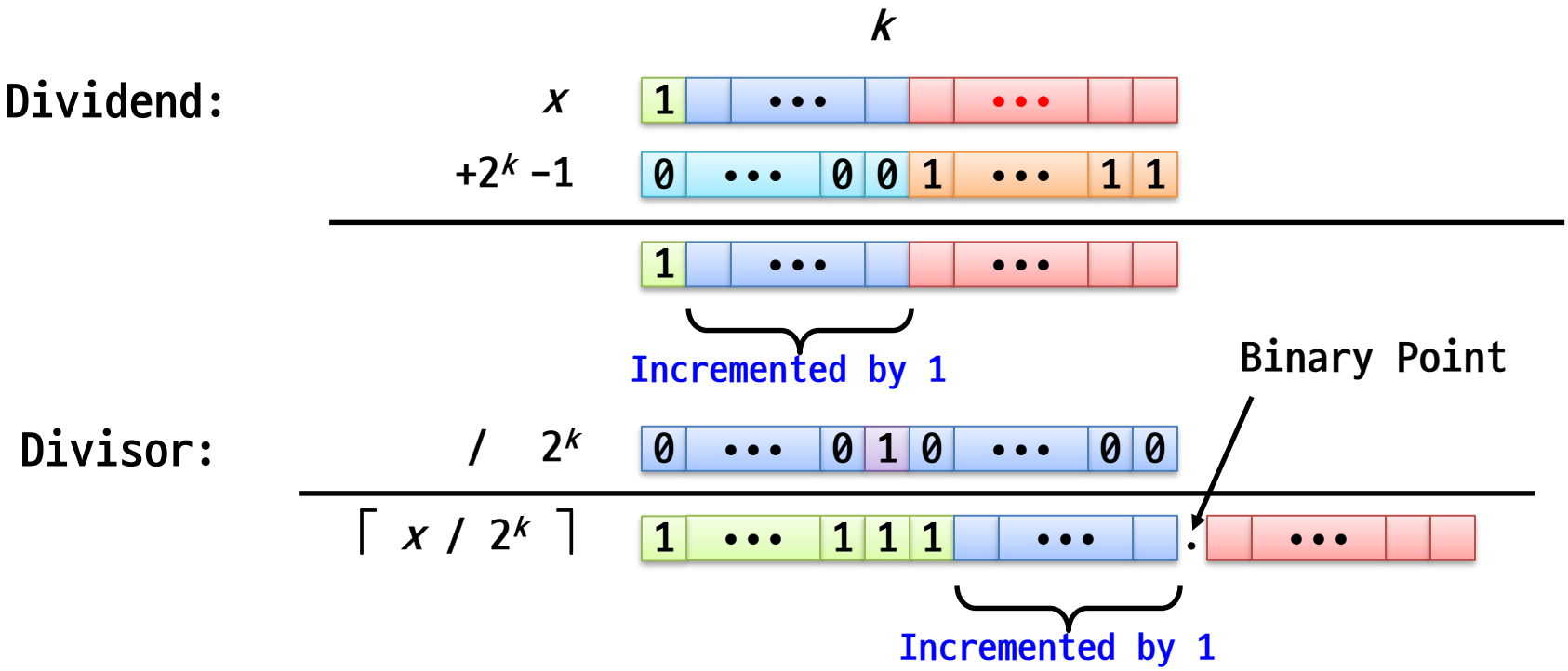
- Biasing has no effect



Division (5)

Case 2: Rounding

- Biasing adds 1 to final result



Division (6)

Compiled signed division code

- Uses arithmetic shift for signed
- Arithmetic shift written as >> in Java

- Compute as $\lceil (x + 2^k - 1) / 2^k \rceil$
  - In C: `(x + (1 << k) - 1) >> k`
  - Biases dividend toward 0

C Function

```
int idiv8 (int  x)
{
   return x / 8;
}
```

Compiled Arithmetic Operations

```
     testl   %eax,  %eax
     js      L4
L3:
     sarl    $3,  %eax
     ret
L4:
     addl    $7,  %eax
     jmp     L3
```

Explanation

```
if  (x  <  0)
   x  += 7;
return  x  >> 3;
```

Division (7)

Division example

```
void main()
{
    char i = -17;

    printf("%d\n", i);

    i = i / 8;
    printf("%d\n", i);

    i = i >> 3;
    printf("%d\n", i);
}
```